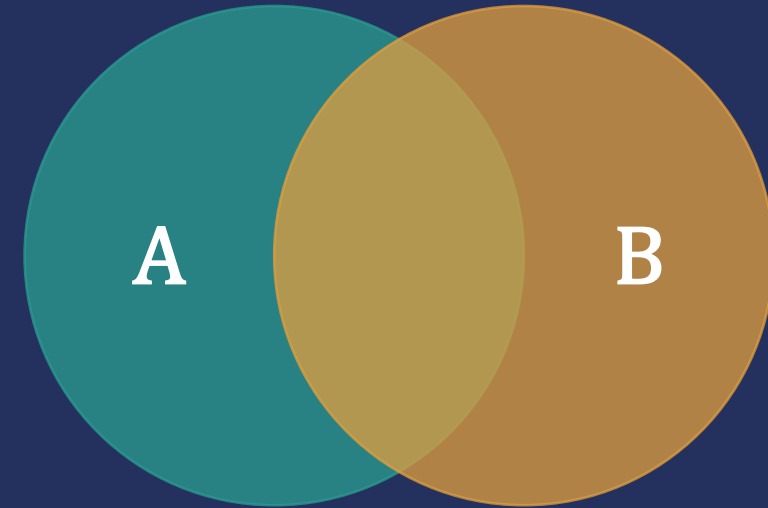


Fuzzy Operations

Union, intersection, complement and beyond, with equations, graphs and worked examples



$\cup$ $\cap$ $\neg$

What you will be able to do



Compute union, intersection and complement of fuzzy sets, both numerically and on a graph



Apply algebraic, bounded and power operations, and build a Cartesian-product relation



Use the algebraic laws (De Morgan included) and explain which crisp laws fail



Solve worked problems with continuous membership functions and linguistic terms

Recap: a fuzzy set assigns degrees

$$A = \{ (x, \mu_A(x)) \mid x \in X \}$$

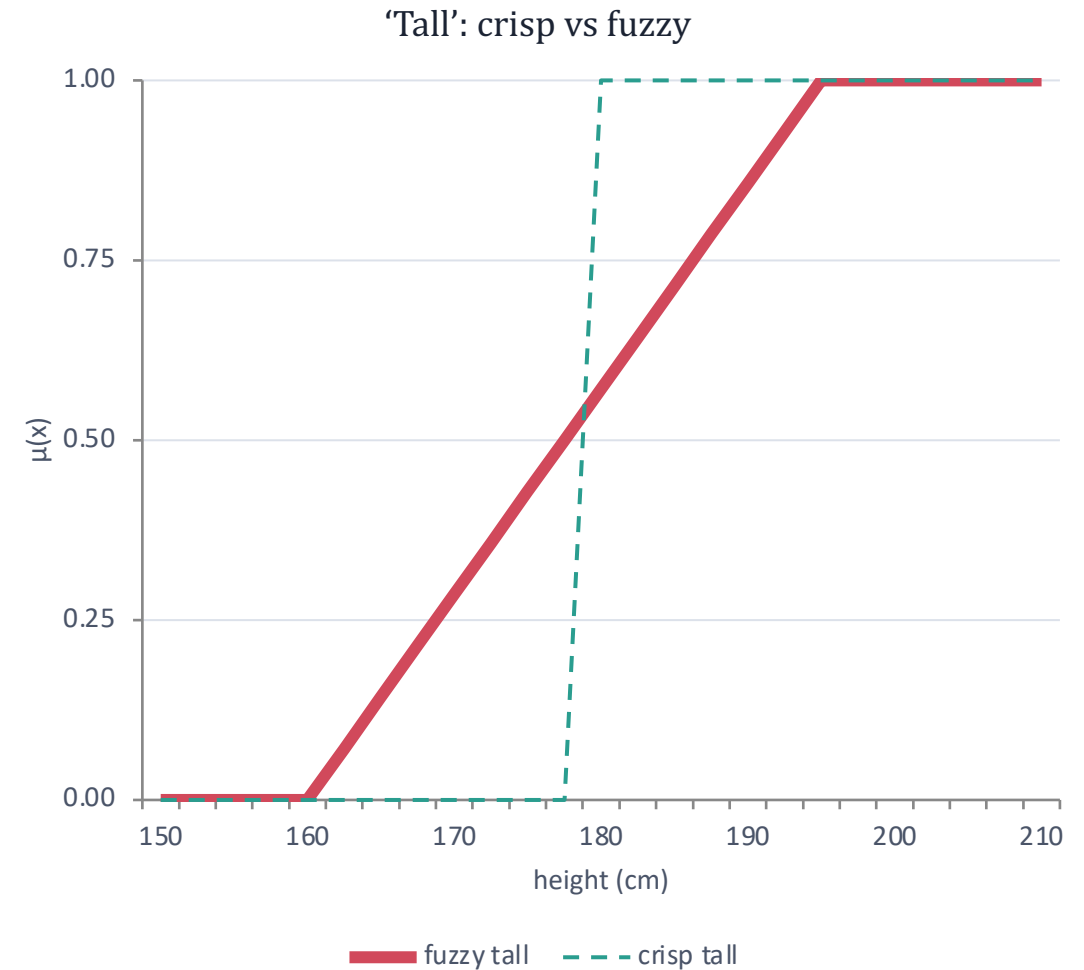
$$\mu_A: X \rightarrow [0,1]$$

Crisp set

Membership is all or nothing: $\mu(x) \in \{0, 1\}$. A 179 cm person is not 'tall', a 180 cm person is.

Fuzzy set

Membership is graded: $\mu(x) \in [0, 1]$. Tallness rises smoothly, so 179 cm and 180 cm differ only a little.



01

Standard operations

Union · Intersection · Complement (Zadeh's max–min–complement)

Union: fuzzy OR takes the maximum

U

Definition

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

Binary operation. Also written $\mu_A(x) \vee \mu_B(x)$.

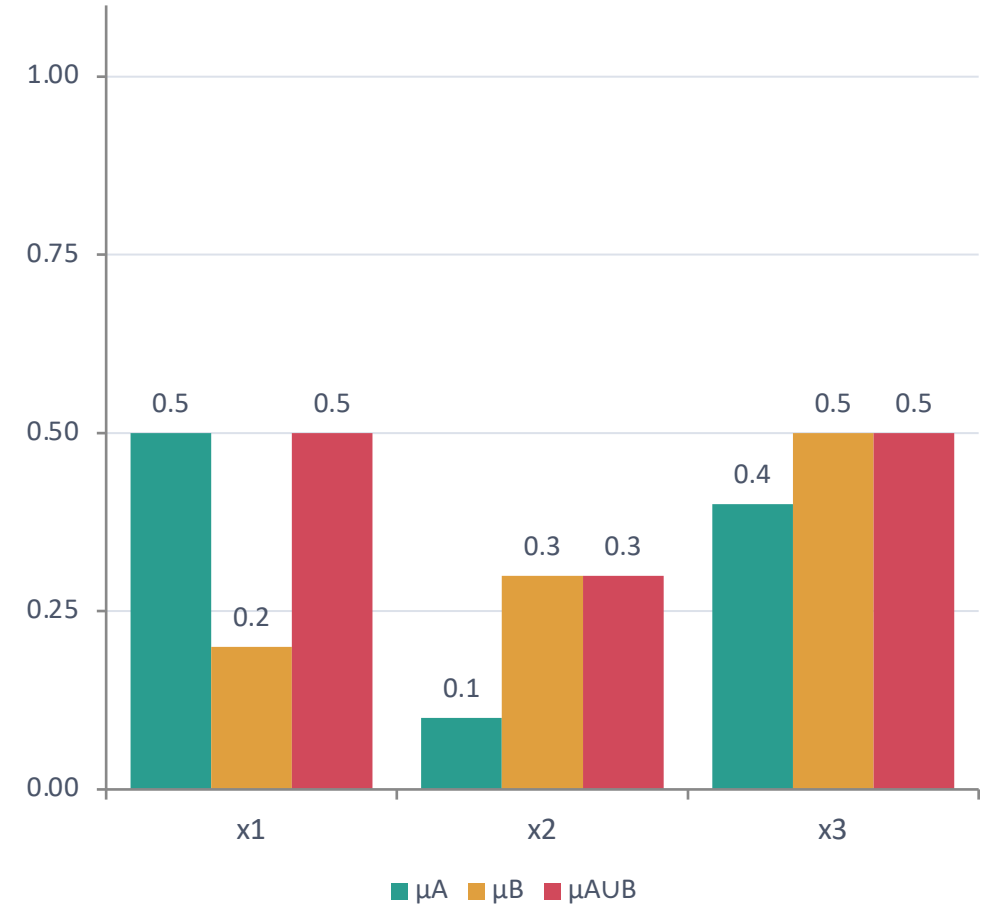
Example

$$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$$

$$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$$

$$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$$

Element by element



Intersection: fuzzy AND takes the minimum



Definition

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

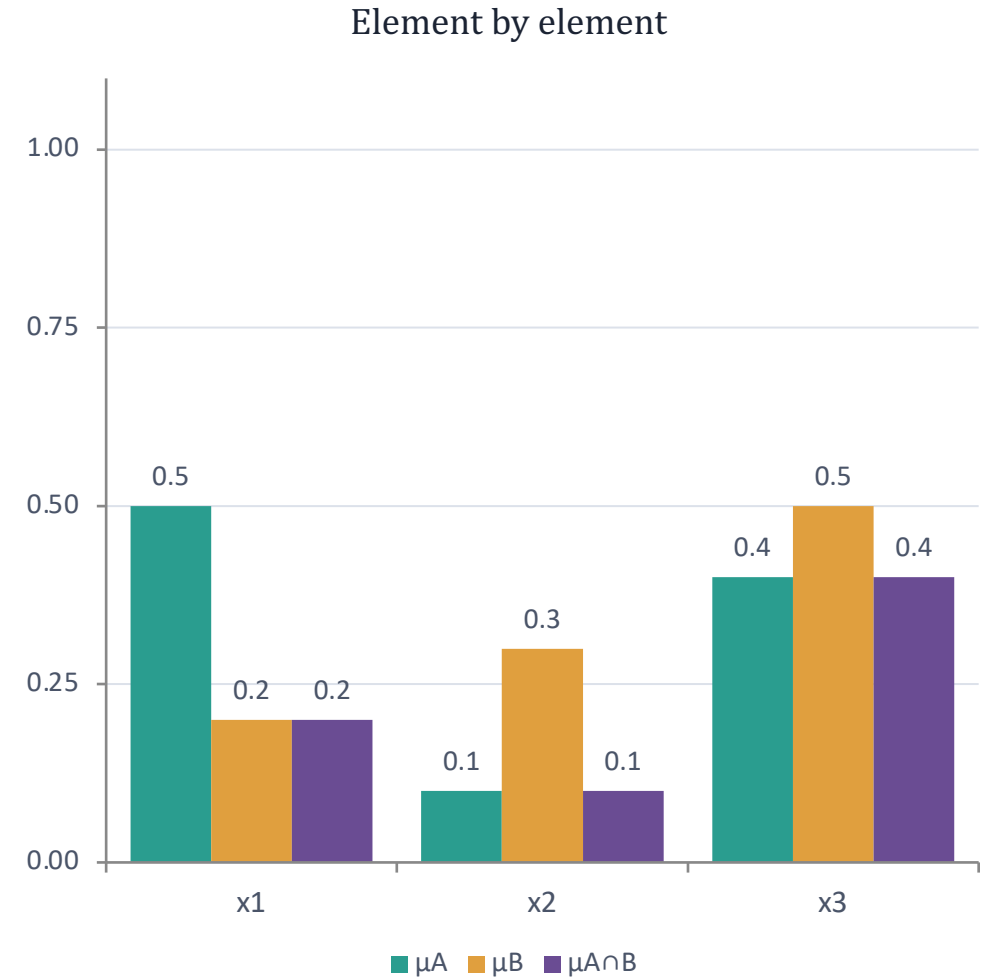
Binary operation. Also written $\mu_A(x) \wedge \mu_B(x)$.

Example

$$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$$

$$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$$

$$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$$



Complement: fuzzy NOT reflects the grade



Definition

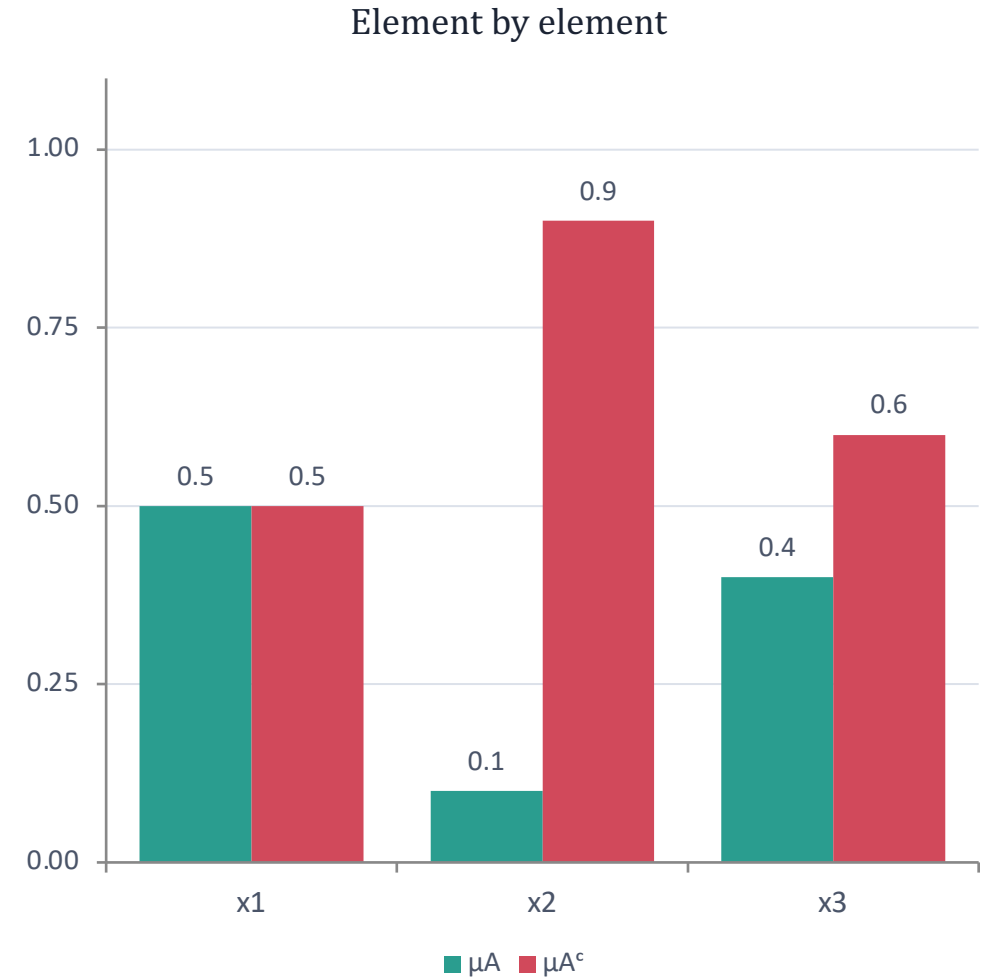
$$\mu_{A^c}(x) = 1 - \mu_A(x)$$

Unary operation: it needs only one fuzzy set.

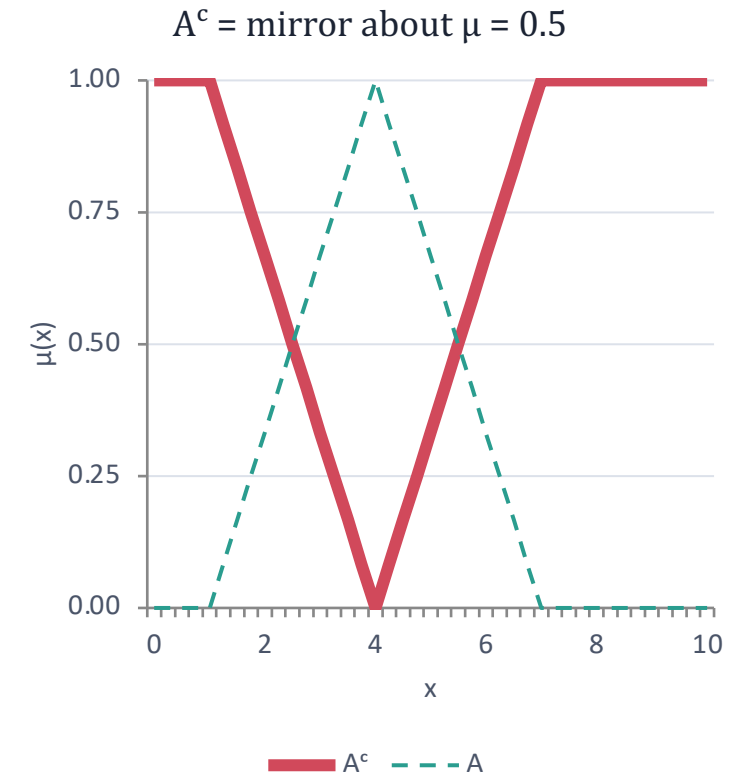
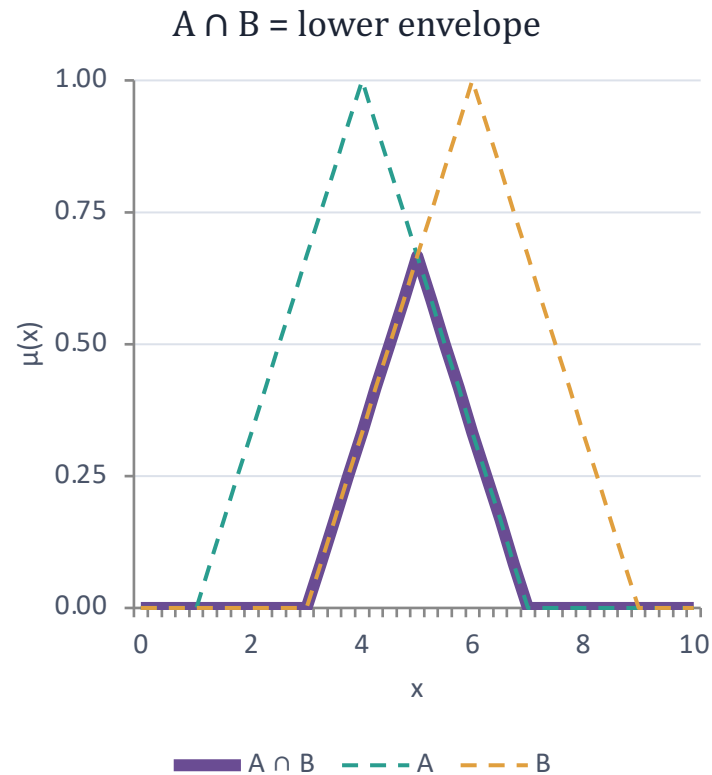
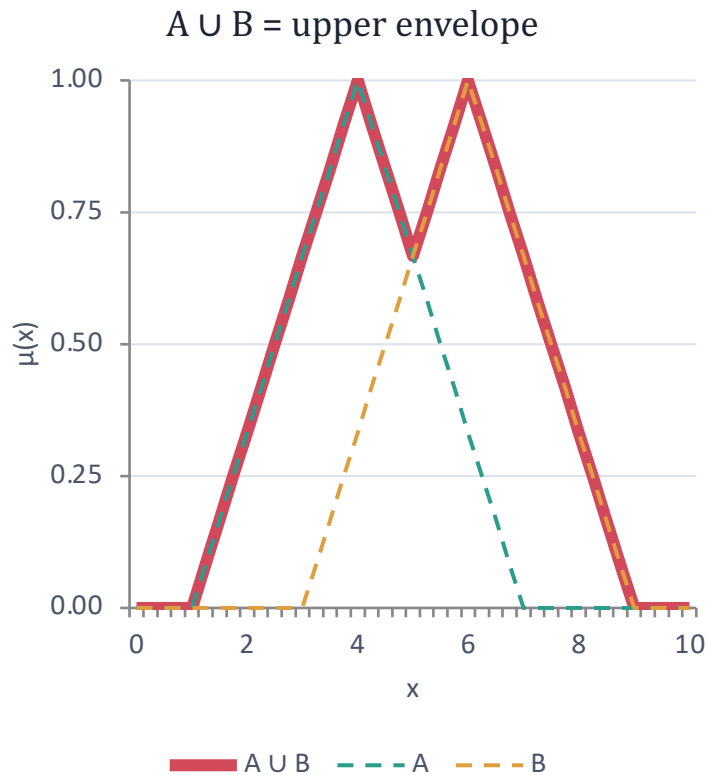
Example

$$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$$

$$C = A^c = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$$



Graphical view: envelopes of the curves



$$A = \text{triangle}(x; 1,4,7), \quad B = \text{triangle}(x; 3,6,9)$$

02

More operations

Products · Sums · Bounded operations · Power · Cartesian product

Algebraic and scalar products



Algebraic (vector) product

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

Example

$$A = \{(x_1, 0.5), (x_2, 0.3)\}, B = \{(x_1, 0.1), (x_2, 0.2)\}$$

$$A \cdot B = \{(x_1, 0.05), (x_2, 0.06)\}$$



Scalar product ($0 \leq \alpha \leq 1$)

$$\mu_{\alpha A}(x) = \alpha \cdot \mu_A(x)$$

Example

$$\alpha = 0.5, A = \{(x_1, 0.5), (x_2, 0.3)\}$$

$$\alpha A = \{(x_1, 0.25), (x_2, 0.15)\}$$

Sum, difference and disjunctive sum



Sum (probabilistic OR)

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \mu_B(x)$$

$x_1: 0.5 + 0.2 - 0.10 = 0.60$



Difference: $A - B = A \cap B^c$

$$\mu_{A-B}(x) = \min(\mu_A(x), 1 - \mu_B(x))$$

$x_1: \min(0.5, 0.8) = 0.5$



Disjunctive sum (fuzzy XOR)

$$A \oplus B = (A^c \cap B) \cup (A \cap B^c)$$

$x_1: \max(\min(0.5, 0.2), \min(0.5, 0.8)) = 0.5$

Bounded sum and bounded difference



Bounded sum

$$\mu_{A \oplus B}(x) = \min\{1, \mu_A(x) + \mu_B(x)\}$$

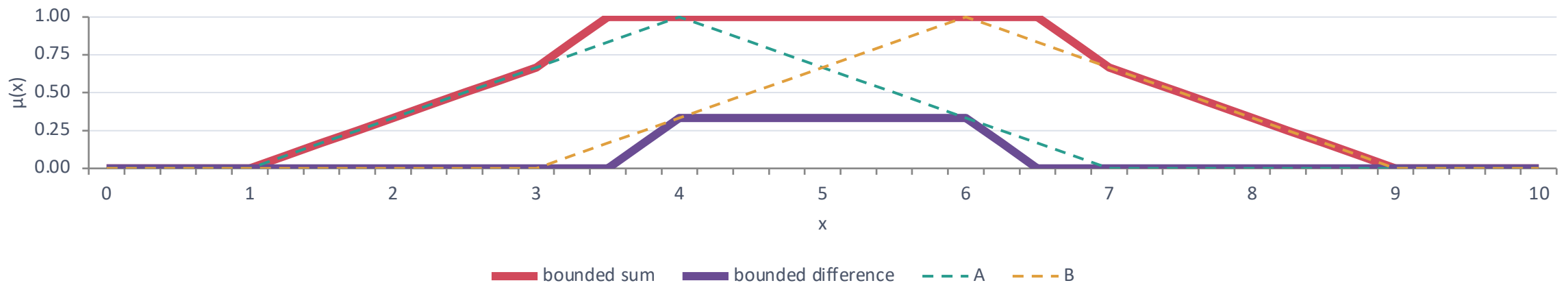
Adds grades but caps the result at 1



Bounded difference

$$\mu_{A \ominus B}(x) = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Positive only where both grades are high



One example, every operation

x	μA	μB	$A \cup B$	$A \cap B$	A^c	$A \cdot B$	$A + B$	$A \oplus B$	$A \ominus B$	$A - B$
x1	0.5	0.2	0.5	0.2	0.5	0.1	0.6	0.7	0	0.5
x2	0.1	0.3	0.3	0.1	0.9	0.03	0.37	0.4	0	0.1
x3	0.4	0.5	0.5	0.4	0.6	0.2	0.7	0.9	0	0.4

- $A \cdot B \leq A \cap B$ and $A + B \geq A \cup B$: the product is a stricter AND, the sum a more generous OR.
- $A \ominus B$ is 0 everywhere here: no element has $\mu A + \mu B > 1$.
- $A - B$ equals A here because every $\mu A \leq 1 - \mu B$.

Equality and power: concentration and dilation

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \quad \forall x \in X$$

$$\mu_{A^\alpha}(x) = (\mu_A(x))^\alpha$$



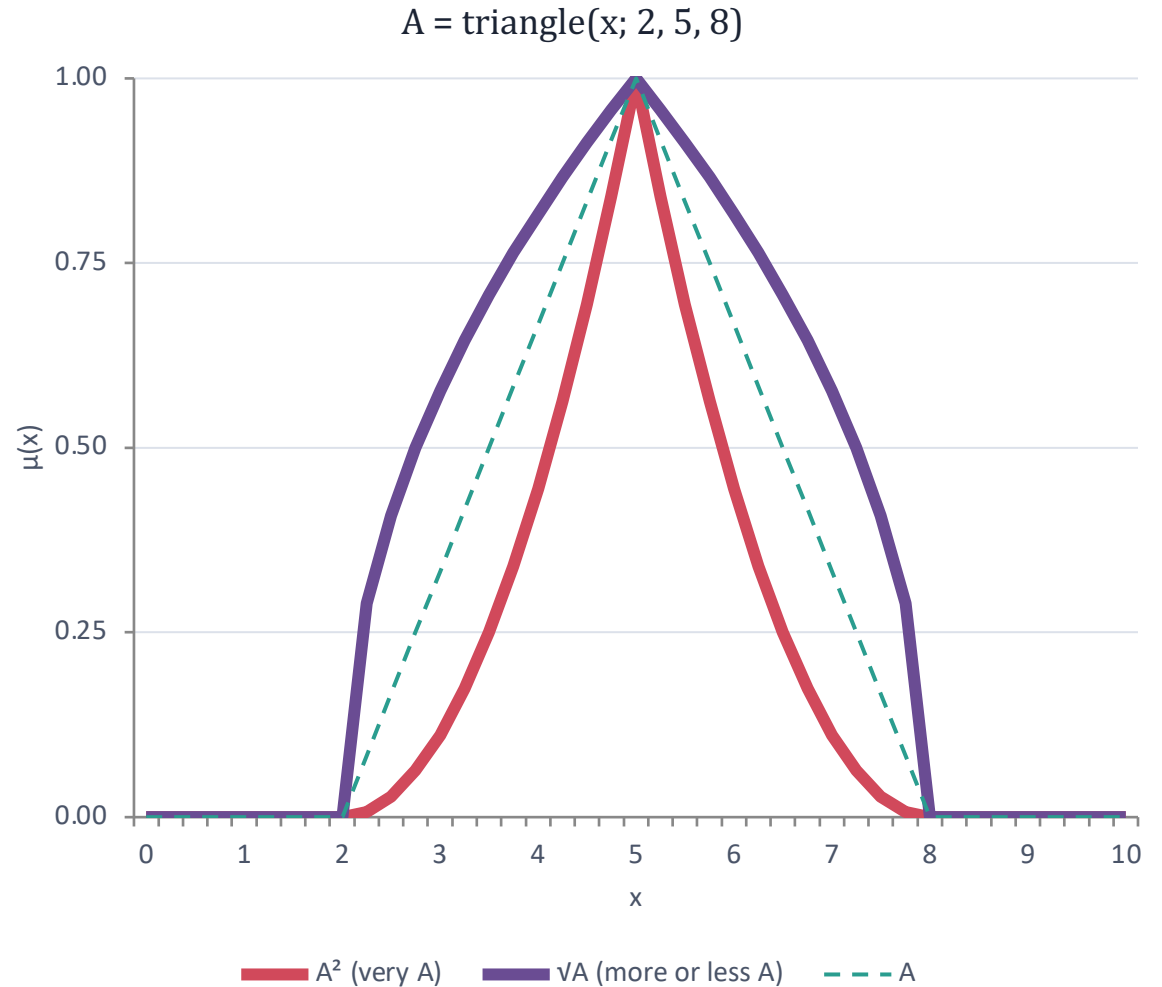
$\alpha > 1$

Concentration: grades shrink, the set gets sharper. $\alpha = 2$ models the hedge 'very'.



$\alpha < 1$

Dilation: grades grow, the set gets wider. $\alpha = 0.5$ models 'more or less'.



Cartesian product builds a fuzzy relation

$$\mu_{A \times B}(x, y) = \min(\mu_A(x), \mu_B(y))$$

$$A = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$$

$$B = \{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$$

- A and B may live on different universes X and Y; union and intersection need the same universe.
- The result is a matrix: the relation matrix $R \subseteq X \times Y$.
- Relations are the basis of fuzzy rules (next lecture).

A × B	y1	y2	y3
x1	0.2	0.2	0.2
x2	0.3	0.3	0.3
x3	0.5	0.5	0.3
x4	0.6	0.6	0.3

Each cell: $\min(\mu_A(x_i), \mu_B(y_j))$. For example x_3, y_3 gives $\min(0.5, 0.3) = 0.3$

03

Properties

Which laws of crisp sets survive, and which do not

Laws that hold (1/2)

Commutativity

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Idempotence

$$A \cup A = A$$
$$A \cap A = A$$

Laws that hold (2/2)

Identity

$$\begin{aligned} A \cup \emptyset &= A, & A \cap X &= A \\ A \cap \emptyset &= \emptyset, & A \cup X &= X \end{aligned}$$

Transitivity

$$A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$$

Involution

$$(A^c)^c = A$$

De Morgan's laws

$$\begin{aligned} (A \cup B)^c &= A^c \cap B^c \\ (A \cap B)^c &= A^c \cup B^c \end{aligned}$$

Laws that fail: excluded middle and contradiction

$$A \cup A^c \neq X \quad A \cap A^c \neq \emptyset$$

Take an element with $\mu_A(x) = 0.5$:

$$\mu_{A \cup A^c}(x) = \max(0.5, 0.5) = 0.5 \neq 1$$

$$\mu_{A \cap A^c}(x) = \min(0.5, 0.5) = 0.5 \neq 0$$

- Fuzzy sets give up the sharp boundary between A and not-A, so they also give up these two laws.
- Bounds that do hold: $\mu_{A \cup A^c} \geq 0.5$ and $\mu_{A \cap A^c} \leq 0.5$.



Beyond max and min: T-norms and S-norms

Any commutative, associative, monotone operator with the right boundary values can act as fuzzy AND (T-norm) or fuzzy OR (S-norm).

T-norm (AND)

S-norm (OR)

Standard

$$\min(a, b)$$

$$\max(a, b)$$

Algebraic

$$a \cdot b$$

$$a + b - ab$$

Bounded

$$\max(0, a + b - 1)$$

$$\min(1, a + b)$$

Drastic

$$\begin{cases} a & b = 1 \\ b & a = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} a & b = 0 \\ b & a = 0 \\ 1 & \text{otherwise} \end{cases}$$

$a = \mu A(x)$, $b = \mu B(x)$. $\min \geq \text{product} \geq \text{bounded} \geq \text{drastic}$, and $\max \leq \text{algebraic sum} \leq \text{bounded sum} \leq \text{drastic sum}$.

04

Worked examples

Continuous membership functions and linguistic terms

Example 1: two continuous fuzzy sets

$$\mu_A(x) = \frac{x}{1+x}, \quad \mu_B(x) = 2^{-x}, \quad x \in [0,5]$$

Find A^c , B^c , $A \cup B$, $A \cap B$ and $(A \cup B)^c$.

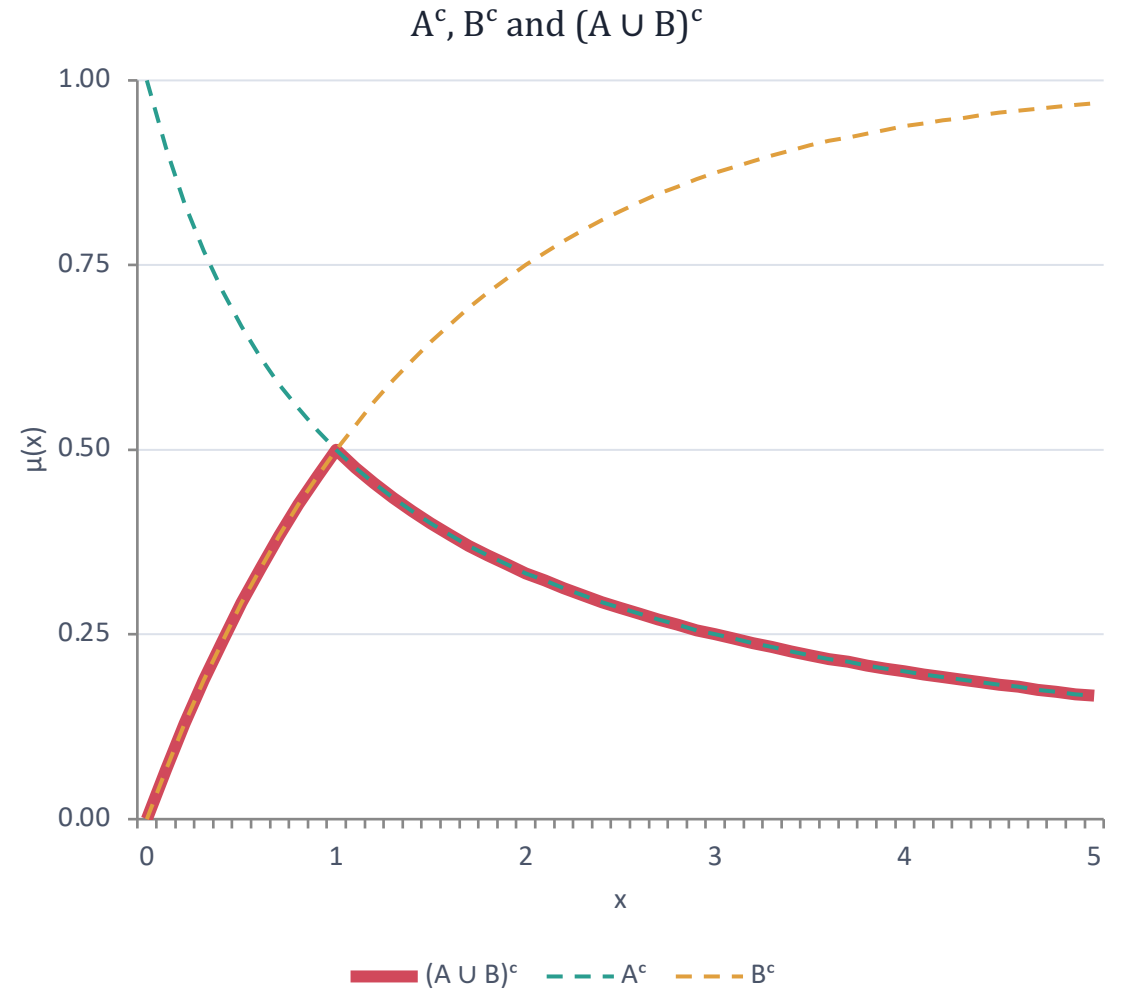
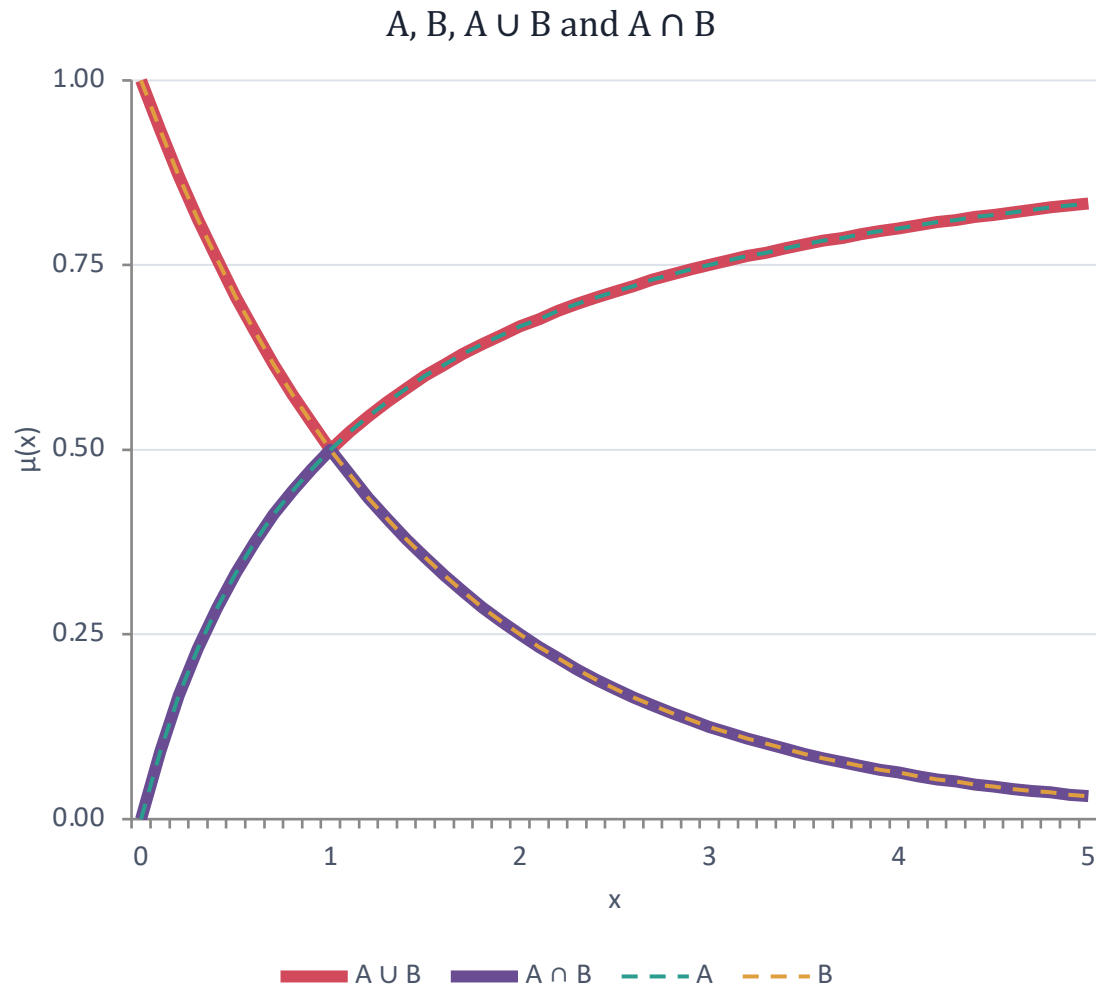
$$\mu_{A^c}(x) = 1 - \frac{x}{1+x} = \frac{1}{1+x}, \quad \mu_{B^c}(x) = 1 - 2^{-x}$$

$$\text{Crossing: } \frac{x}{1+x} = 2^{-x} \text{ at } x = 1 \quad (1/2 = 2^{-1})$$

$$\mu_{A \cup B}(x) = \begin{cases} 2^{-x} & 0 \leq x \leq 1 \\ \frac{x}{1+x} & 1 \leq x \leq 5 \end{cases} \quad \mu_{A \cap B}(x) = \begin{cases} \frac{x}{1+x} & 0 \leq x \leq 1 \\ 2^{-x} & 1 \leq x \leq 5 \end{cases}$$

$$\mu_{(A \cup B)^c}(x) = \mu_{A^c \cap B^c}(x) = \begin{cases} 1 - 2^{-x} & 0 \leq x \leq 1 \\ \frac{1}{1+x} & 1 \leq x \leq 5 \end{cases}$$

Example 1: the graphs



Example 2: operations carry meaning

A = cold climate, B = hot climate (temperature t in $^{\circ}\text{C}$)

Not cold

$$\bar{A}: 1 - \mu_A(t)$$

Not hot

$$\bar{B}: 1 - \mu_B(t)$$

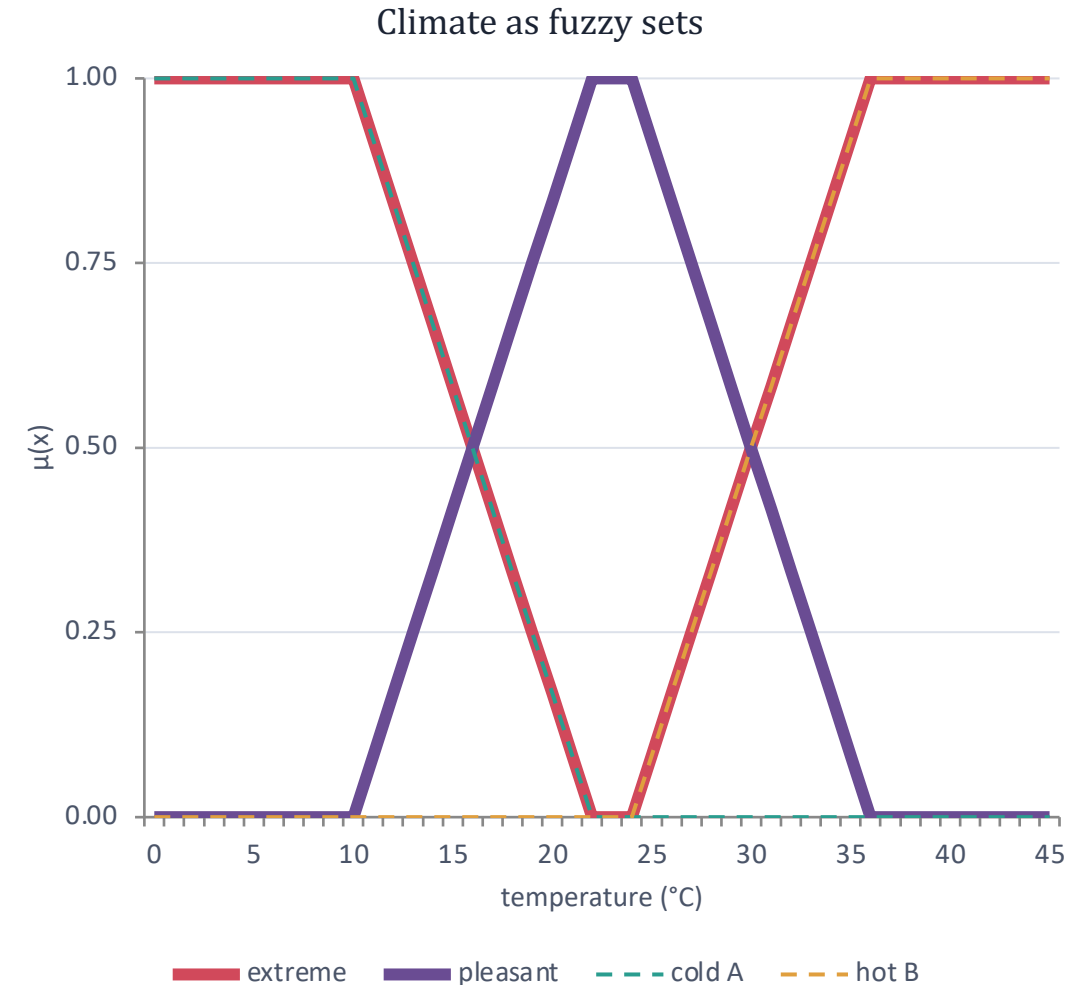
Extreme

$$A \cup B: \max(\mu_A, \mu_B)$$

Pleasant

$$\bar{A} \cap \bar{B} = (A \cup B)^c$$

“Not cold” is not the same as “hot”: at 16 $^{\circ}\text{C}$, not cold = 0.5 but hot = 0.



Summary

Union	$\max(\mu_A, \mu_B)$	Intersection	$\min(\mu_A, \mu_B)$
Complement	$1 - \mu_A$	Algebraic product	$\mu_A \cdot \mu_B$
Sum	$\mu_A + \mu_B - \mu_A \mu_B$	Difference	$\min(\mu_A, 1 - \mu_B)$
Bounded sum	$\min(1, \mu_A + \mu_B)$	Bounded difference	$\max(0, \mu_A + \mu_B - 1)$
Power	$(\mu_A)^\alpha$	Cartesian product	$\min(\mu_A(x), \mu_B(y))$

- Every operation acts element by element on membership grades and returns a fuzzy set.
- Commutative, associative, distributive, idempotent, identity, involution and De Morgan laws hold.
- Excluded middle and contradiction fail: $A \cup A^c \neq X$ and $A \cap A^c \neq \emptyset$.

05

Questions

Review questions and practice problems

Review questions

1

Why is the complement a unary operation while union and intersection are binary?

2

Explain why the laws of excluded middle and contradiction fail for fuzzy sets. Give a numerical example.

3

Distinguish concentration from dilation and link each to a linguistic hedge.

4

How does the Cartesian product differ from union and intersection in terms of the universe of discourse?

5

State both De Morgan laws and verify one for $\mu_A = 0.7$, $\mu_B = 0.4$.

6

What is a T-norm? Give three T-norms and their dual S-norms.

Practice problems

$$A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1.0), (x_4, 0.4)\}, \quad B = \{(x_1, 0.6), (x_2, 0.3), (x_3, 0.5), (x_4, 0.9)\}$$
$$C = \{(y_1, 0.8), (y_2, 0.4)\}$$

7

Find $A \cup B$, $A \cap B$, A^c and B^c .

8

Find $A \cdot B$, $A + B$, $A \oplus B$ and $A \ominus B$.

9

Find $A - B$ and the disjunctive sum.

10

Verify $(A \cup B)^c = A^c \cap B^c$.

11

Compute A^2 and $\forall A$. Which one is 'very A'?

12

Build the relation matrix $A \times C$.

13

For $\mu_A = x/(1+x)$, $\mu_B = 2^{-x}$ find $\mu_{A \cup B}(2)$ and $\mu_{A \cap B}(0.5)$.

14

Sketch $A = \text{triangle}(x; 0, 3, 6)$ and $B = \text{triangle}(x; 2, 5, 8)$ with their union and intersection.

References

- D. Samanta, Introduction to Soft Computing, NPTEL, IIT Kharagpur, Lecture 4: Fuzzy operations.
- 3G Editorial Board, Illustrated Handbook of Soft Computing, Chapter 'Fuzzy Systems': operations on fuzzy sets and their properties, pp. 86–88.
- J. R. Raol and S. M. George, Intelligent Estimation: A Soft Computing Paradigm, §5.3 Multivariate logic: fuzzy logic operators, T-norms and S-norms.
- L. A. Zadeh, “Fuzzy sets,” Information and Control, vol. 8, pp. 338–353, 1965.